

Yang-Coulomb Monopole

Levon Mardoyan¹

Yerevan State University
International Center for Advanced Studies
Alex Manougian str., 1, 375025 Yerevan, Armenia

Abstract

In this article, in the framework of an analytical approach and with help of the generalized version of the Hurwitz transformation the five-dimensional bound system composed of the Yang monopole coupled to a particle of the isospin by $SU(2)$ and Coulomb interaction is constructed from the eight-dimensional quantum oscillator. The generalized Runge-Lenz vector and the $SO(6)$ group of the hidden symmetry are established. It is also shown that group of hidden symmetry makes it possible to calculate the spectrum of system by a pure algebraic method.

1 Introduction

The objective of the present work is to illustrate the property of the Schrödinger equation which is called here the dyon-oscillator duality. The property is in the following. The Schrödinger equation for an oscillator possesses two parameters – the energy E and the cyclic frequency ω . The quantization leads to the constraint $E = \hbar\omega(N + D/2)$, where $N = 0, 1, 2, \dots$, and D is the dimension of the configuration space of the oscillator. If ω is fixed then E is quantized, and that is the standard situation. Imagine for a moment that now E is fixed. Whence, necessarily ω is quantized, and we are in nonstandard situation. The question is whether the nonstandard situation corresponds to any physics, i.e. is it possible to find such a transformation that converts the oscillator to some physical system with coupling constant α , being a function of E , and energy ε , depending on ω ? If there exists such a transformation, we can confirm that the "nonstandard oscillator" is identical to that physical system. Below will be shown the validity of the described picture for dimensions $D = 1, 2, 4, 8$, and that the final system is bound system of charge-particle (remind, that dyon is the hypothetical particle introduced by Schwinger [1], which is unlike the Dirac monopole endowed with not just magnetic but electric charge as well). As the "standard" and "nonstandard" regimes are mutually exclusive, the initial oscillator and the final "charge-dyon" system are dual to each other, and that explains the relevancy of the term "dyon-oscillator duality". Note also, that in the initial system the spectrum is discrete only, i.e. the particle has just a finite motion (for such cases it is used to say that we have a model with confinement). Generally speaking, the spectrum of the final system includes the discrete spectrum as well as the continuous one, i.e. in that model there is no confinement. However, unlike the first model, in the second model we have monopoles. There is some analogy between the dyon-oscillator and the Seiberg-Witten

¹E-mail: mardoyan@icas.yasu.am

duality [2], according to which the gauge theories with strong interactions are equivalent to the theories having weak interaction on one hand and topological nontrivial objects, such as monopoles and dyons are, on the other hand.

2 Coulomb-Oscillator Duality

Let us consider the equation

$$\frac{d^2 R}{du^2} + \frac{D-1}{u} \frac{dR}{du} - \frac{L(L+D-2)}{u^2} R + \frac{2m}{\hbar^2} \left(E - \frac{m\omega^2 u^2}{2} \right) R = 0. \quad (1)$$

Here R is the radial part of the wave function for D -dimensional oscillator ($D > 2$), $L = 0, 1, 2, \dots$ are the eigenvalues of the global angular momentum.

After substitution $r = u^2$ the equation (1) transforms to the equation

$$\frac{d^2 R}{dr^2} + \frac{d-1}{r} \frac{dR}{dr} - \frac{l(l+d-2)}{r^2} R + \frac{2m}{\hbar^2} \left(\epsilon + \frac{e^2}{r} \right) R = 0, \quad (2)$$

where $d = D/2 + 1$, $l = L/2$,

$$\epsilon = -m\omega^2/8, \quad e^2 = E/4. \quad (3)$$

This is quite an unexpected result. If $D = 4, 6, 8, \dots$, then $d = 3, 4, 5, \dots$, and the equation (2) is formally identical to the radial equation for d -dimensional Coulomb problem (for odd $D > 2$ the value of d is half-integer and so cannot have the meaning of the dimension of the space in the usual sense). Then, l takes not only integer but half-integer values as well, hence it has a meaning of general momentum and a question arise about the origin of the fermion degree of freedom. The answer to the question will be given later. Finally, as was mentioned in first section, the equations (1) and (2) are dual to each other and the duality transformation is $r = u^2$.

Up to now, just the radial part of the wave function of the oscillator has been considered. For the Schrödinger equation we must take into account the angular part as well. Thus, the duality transformation must also include the transformation of angular variables. If we interpret the change of variables $r = u^2$ as the mechanism of generation of electric charge, (as will be shown later) the transformation of some angular variables is responsible for the generation of magnetic charges.

Therefore, we return to the condition $r = u^2$. In the Cartesian coordinates this condition has the form

$$x_0^2 + x_1^2 + \dots + x_{d-1}^2 = \left(u_0^2 + u_1^2 + \dots + u_{D-1}^2 \right)^2,$$

which is called the Euler's identity. According to Hurwitz theorem [3] this equation has a solution bilinear in u_μ only for the following pairs of numbers

$$(D, d) = (1, 1); (2, 2); (4, 3); (8, 5).$$

Three remarkable circumstances are connected with the last relation:

- $(D, d) = (2, 2)$ – is the Levi-Civita transformation [4].
 $(D, d) = (4, 3)$ – is the Kustaanheimo-Stiefel transformation [5].
 $(D, d) = (8, 5)$ – is the Hurwitz transformation [6,7,8].
- The transformation (4) establishes the connection between two fundamental problems of mechanics, the oscillator and Kepler problems.
- The "magic" numbers $D = 1, 2, 4, 8$ have the direct relation to the existence fact of four basic algebraic structures: real numbers, complex numbers, quaternions and octanions.

Later on we consider only the case $(D, d) = (8, 5)$.

3 Hurwitz Transformation

We can write the solution of the Euler's identity as

$$x = H(u; D)u, \quad (4)$$

which can be interpreted as a bilinear transformation that maps one Euclidean space into another. Here D is the dimension of the space, H is the matrix $D \times D$ with the elements u_μ , and x, u are D -dimensional columns composed from x_j, u_μ and, possibly, zeroes. So, for the Levi-Civita and Kustaanheimo-Stiefel transformations we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_3 & -u_4 & u_1 & -u_2 \\ u_4 & u_3 & u_2 & u_1 \\ u_1 & u_2 & -u_3 & -u_4 \\ u_2 & -u_1 & -u_4 & u_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

The matrixes $H(u; 2)$ and $H(u; 4)$ have the property

$$H(u; 2) H^T(u; 2) = u^2 E(2), \quad H(u; 4) H^T(u; 4) = u^2 E(4),$$

where " T " means the sign of the transposition, $E(2)$ and $E(4)$ are the unit matrixes. Due to these properties the Euler identities are fulfilled. Now it is easily deduced, that the transformation $\mathbb{R}^8(\vec{u}) \rightarrow \mathbb{R}^5(\vec{x})$ must take the form [6]

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_0 & u_1 & u_2 & u_3 & -u_4 & -u_5 & -u_6 & -u_7 \\ u_4 & u_5 & -u_6 & -u_7 & u_0 & u_1 & -u_2 & -u_3 \\ u_5 & -u_4 & u_7 & -u_6 & -u_1 & u_0 & -u_3 & u_2 \\ u_6 & u_7 & u_4 & u_5 & u_2 & u_3 & u_0 & u_1 \\ u_7 & -u_6 & -u_5 & u_4 & u_3 & -u_2 & -u_1 & u_0 \\ u_1 & -u_0 & u_3 & -u_2 & u_5 & -u_4 & u_7 & -u_6 \\ u_2 & -u_3 & -u_0 & u_1 & -u_6 & u_7 & u_4 & -u_5 \\ u_3 & u_2 & -u_1 & -u_0 & -u_7 & -u_6 & u_5 & u_4 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix}.$$

Hence it follows that

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2 - u_6^2 - u_7^2, \\
x_1 &= 2(u_0u_4 - u_1u_5 - u_2u_6 - u_3u_7), \\
x_2 &= 2(u_0u_5 + u_1u_4 - u_2u_7 + u_3u_6), \\
x_3 &= 2(u_0u_6 + u_1u_7 + u_2u_4 - u_3u_5), \\
x_4 &= 2(u_0u_7 - u_1u_6 + u_2u_5 + u_3u_4).
\end{aligned} \tag{5}$$

It is easy to prove that for the matrix $H(u; 8)$ there is a condition

$$H(u; 8)H^T(u; 8) = u^2E(8)$$

that guarantee the validity of Euler identity.

Adding to (5) the transformations [9]

$$\begin{aligned}
\alpha_T &= \frac{i}{2} \ln \frac{(u_0 + iu_1)(u_2 - iu_3)}{(u_0 - iu_1)(u_2 + iu_3)} \in [0, 2\pi), \\
\beta_T &= 2 \arctan \left(\frac{u_0^2 + u_1^2}{u_2^2 + u_3^2} \right)^{1/2} \in [0, \pi], \\
\gamma_T &= \frac{i}{2} \ln \frac{(u_0 - iu_1)(u_2 - iu_3)}{(u_0 + iu_1)(u_2 + iu_3)} \in [0, 4\pi),
\end{aligned} \tag{6}$$

we obtain a transformation converting \mathbb{R}^8 to the direct product $\mathbb{R}^5 \otimes \mathbf{S}^3$ of the space $\mathbb{R}^5(\vec{x})$ and a three-dimensional sphere $\mathbf{S}^3(\alpha_T, \beta_T, \gamma_T)$.

4 Dyon-Oscillator Duality

In the coordinates (5), (6) we can transform the Schrödinger equation for the eight-dimensional isotropic oscillator

$$\frac{\partial^2 \psi}{\partial u_\mu^2} + \frac{2m}{\hbar^2} \left(E - \frac{m\omega^2 u^2}{2} \right) \psi = 0, \quad u_\mu \in R^8$$

into the equation [10]

$$\frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial x_j} - \hbar A_j^a \hat{T}_a \right)^2 \psi + \frac{\hbar^2}{2mr^2} \hat{T}^2 \psi - \frac{e^2}{r} \psi = \epsilon \psi \tag{7}$$

where ϵ and e^2 are defined by the relation (3). The operators \hat{T}_a are the generators of the $SU(2)$ group. In coordinates $(\alpha_T, \beta_T, \gamma_T)$ they are parametrized as follows

$$\hat{T}_1 = i \left(\cos \alpha_T \cos \beta_T \frac{\partial}{\partial \alpha_T} + \sin \alpha_T \frac{\partial}{\partial \beta_T} - \frac{\cos \alpha_T}{\sin \beta_T} \frac{\partial}{\partial \gamma_T} \right),$$

$$\begin{aligned}\hat{T}_2 &= i \left(\sin \alpha_T \cot \beta_T \frac{\partial}{\partial \alpha_T} - \cos \alpha_T \frac{\partial}{\partial \beta_T} - \frac{\sin \alpha_T}{\sin \beta_T} \frac{\partial}{\partial \gamma_T} \right), \\ \hat{T}_3 &= -i \frac{\partial}{\partial \alpha_T}.\end{aligned}$$

Recall that the operators \hat{T}_a satisfy the following commutation relations

$$[\hat{T}_a, \hat{T}_b] = i \varepsilon_{abc} \hat{T}_c.$$

The five-dimensional vectors \vec{A}^a are given by the expressions

$$\begin{aligned}\vec{A}^1 &= \frac{1}{r(r+x_0)} (0, x_4, x_3, -x_2, -x_1), \\ \vec{A}^2 &= \frac{1}{r(r+x_0)} (0, -x_3, x_4, x_1, -x_2), \\ \vec{A}^3 &= \frac{1}{r(r+x_0)} (0, x_2, -x_1, x_4, -x_3),\end{aligned}$$

The vectors \vec{A}^a are orthogonal to each other,

$$A_j^a A_j^b = \frac{1}{r^2} \frac{r-x_0}{r+x_0} \delta_{ab}$$

and to the vector $\vec{x} = (x_0, x_1, x_2, x_3, x_4)$ as well.

The equation (7) is identical to the Pauli equation and therefore we can give to the triplet of five-dimensional vectors \vec{A}^a the meaning of the vector potentials with the line of singularity along the nonpositive x_0 semiaxis.

The five-dimensional vector potentials \vec{B}^a

$$\begin{aligned}\vec{B}^1 &= \frac{1}{r(r-x_0)} (0, -x_4, x_3, -x_2, x_1), \\ \vec{B}^2 &= \frac{1}{r(r-x_0)} (0, -x_3, -x_4, x_1, x_2), \\ \vec{B}^3 &= \frac{1}{r(r-x_0)} (0, x_2, -x_1, -x_4, x_3)\end{aligned}$$

with the singularity axis, directed along the nonnegative x_0 semiaxis, are obtained from the vectors \vec{A}^a by the following gauge transformation

$$B_j = \hat{S} A_j \hat{S}^{-1} + i \hat{S} \frac{\partial}{\partial x_j} \hat{S}^{-1}.$$

Here $A_j = A_j^a \hat{T}_a$, $B_j = B_j^a \hat{T}_a$, and

$$\hat{S} = e^{-i\gamma \hat{T}_3} e^{-i\beta \hat{T}_2} e^{-i\alpha \hat{T}_1}.$$

The hyperspherical angles α , β and γ are defined as

$$\begin{aligned}\alpha &= \frac{i}{2} \ln \frac{(x_2 - ix_1)(x_4 - ix_3)}{(x_2 + ix_1)(x_4 + ix_3)} \in [0, 2\pi) \\ \beta &= 2 \arctan \left(\frac{x_1^2 + x_2^2}{x_3^2 + x_4^2} \right)^{1/2} \in [0, \pi] \\ \gamma &= \frac{i}{2} \ln \frac{(x_2 + ix_1)(x_4 - ix_3)}{(x_2 - ix_1)(x_4 + ix_3)} \in [0, 4\pi).\end{aligned}$$

Now, it is necessary to explain what the physical system the equation (7) describes.

5 Field Tensor

For the first step we rewrite the five-dimensional vector potentials \vec{A}^a in the following form

$$A_i^a = \frac{2ig}{r(r+x_0)} \tau_{ij}^a x_j. \quad (8)$$

Here τ_{ij}^a are the 5×5 matrices having the following explicit form

$$\tau^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\sigma^1 \\ 0 & i\sigma^1 & 0 \end{pmatrix}, \quad \tau^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sigma^3 \\ 0 & -i\sigma^3 & 0 \end{pmatrix}, \quad \tau^3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix},$$

where σ^a are the Pauli matrices and the τ^a matrices satisfying $[\tau^a, \tau^b] = i\epsilon_{abc}\tau^c$. It is obvious that $\tau_{ij}^a = -\tau_{ji}^a$ and, therefore, the vectors \vec{A}^a are orthogonal to \vec{x} . Moreover, for the τ^a matrices the following relations occur

$$4\tau_{ij}^a \tau_{jk}^b = \delta_{ab} (\delta_{ik} - \delta_{i0}\delta_{k0}) + 2i\epsilon_{abc}\tau_{ik}^c, \quad (9)$$

$$\begin{aligned}\epsilon_{abc}\tau_{ij}^b \tau_{km}^c &= \frac{i}{2} \left[(\delta_{i0}\delta_{k0} - \delta_{ik}) \tau_{jm}^a - (\delta_{i0}\delta_{m0} - \delta_{im}) \tau_{jk}^a + \right. \\ &\quad \left. + (\delta_{j0}\delta_{m0} - \delta_{jm}) \tau_{ik}^a - (\delta_{j0}\delta_{k0} - \delta_{jk}) \tau_{im}^a \right].\end{aligned} \quad (10)$$

Now, using the definition of the Yang-Mills field tensor

$$F_{ij}^a = \frac{\partial A_j^a}{\partial x_i} - \frac{\partial A_i^a}{\partial x_j} + \epsilon_{abc} A_i^{b(+)} A_j^{c(+)}$$

and the expressions (8) and (10) we can write the field tensor F_{ij}^a in a more explicit form

$$F_{ij}^a = \frac{1}{r^2} \left[(x_j + r\delta_{j0}) A_i^a - (x_i + r\delta_{i0}) A_j^a - 2i\tau_{ij}^a \right].$$

The straightforward computation gives

$$F_{ij}^a F_{jk}^b = \frac{1}{r^6} (x_i x_k - r^2 \delta_{ik}) \delta_{ab} + \frac{1}{r^2} \epsilon_{abc} F_{ik}^c.$$

or

$$F_{ij}^a F_{ij}^b = \frac{4}{r^4} \delta_{ab}. \quad (11)$$

6 Topological Charge

It is convenient to perform the computation of the topological charge in the five-dimensional hyperspherical coordinates, which we define as

$$\begin{aligned} x_0 &= r \cos \theta, \\ x_2 + ix_1 &= r \sin \theta \sin \frac{\beta}{2} e^{i \frac{\alpha - \gamma}{2}}, \\ x_4 + ix_3 &= r \sin \theta \cos \frac{\beta}{2} e^{i \frac{\alpha + \gamma}{2}}, \end{aligned}$$

where $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$, $\gamma \in [0, 4\pi)$.

It is known that the components of the second rank tensor in the different coordinate systems are interrelated by the formula

$$\bar{f}_{ik} = \frac{\partial x_m}{\partial \bar{x}_i} \frac{\partial x_n}{\partial \bar{x}_k} f_{mn}$$

in our case $\bar{x}_0 = r$, $\bar{x}_1 = \theta$, $\bar{x}_2 = \beta$, $\bar{x}_3 = \alpha$, $\bar{x}_4 = \gamma$. The direct computation gives that $F_{rk}^a \equiv 0$, where $k = r, \theta, \beta, \alpha, \gamma$ (the other components see in the Appendix).

Using the explicit forms of the hyperspherical components F_{ij}^a it is possible to verify that the field tensor F_{ij}^a is a self-duality

$$*F^{a(+)\mu\nu} = F^{a(+)\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \quad (12)$$

Further, using the definition of the topological charge

$$q = \frac{1}{32\pi^2} \sum_{a=1}^3 \oint *F^{a(+)\mu\nu} F_{\mu\nu}^a dS$$

where $dS = \frac{r^4}{8} \sin^3 \theta \sin \beta d\theta d\beta d\alpha d\gamma$, and taking into account equations of the self-duality (12) and orthogonality conditions (11) we obtain that in our case $q = +1$.

So, we see that the equation (7) describe the charge-dyon system with $SU(2)$ Yang monopole [11], and Schrödinger equation for the eight-dimensional isotropic oscillator and equation (7) are dual to each other.

It is important to note the following fact

$$q^a = \frac{1}{32\pi^2} \oint *F^{a(+)\mu\nu} F_{\mu\nu}^a dS = \frac{1}{3},$$

i.e. 1/3 topological charge correspond to each a -th component of the gauge field F_{ij}^a .

7 Hidden Symmetry

Since, our obtained system is a non-Abelian extension of the five-dimensional Coulomb problem, it is natural to try to construct an analog of the Runge-Lenz vector for Yang-Coulomb monopole by passing from $\mathbb{R}^3(\vec{x})$ to $\mathbb{R}^5(\vec{x})$ and taking account of the gauge field [12]. The first step was made many years ago [13]

$$\hat{M}_k = \frac{1}{2\sqrt{\mu_0}} \left(\hat{p}_i \hat{l}_{ik} + \hat{l}_{ik} \hat{p}_i + \frac{2\mu_0 e^2}{\hbar} \frac{x_k}{r} \right),$$

where $\hat{p}_i = -i\hbar \partial / \partial x_i$, $\hat{l}_{ij} = \frac{1}{\hbar} (x_i \hat{p}_j - x_j \hat{p}_i)$. The second step can be realized by the substitution [11]:

$$\begin{aligned} \hat{p}_i &\rightarrow \hat{\pi}_i = \hat{p}_i - \hbar A_i^a \hat{T}_a, \\ \hat{l}_{ij} &\rightarrow \hat{L}_{ij} = \frac{1}{\hbar} (x_i \hat{\pi}_j - x_j \hat{\pi}_i) - r^2 F_{ij}^a \hat{T}_a. \end{aligned}$$

The following fundamental commutation relations are valid

$$[\hat{\pi}_i, x_j] = -i\hbar \delta_{ij}, \quad [\hat{\pi}_i, \hat{\pi}_j] = i\hbar^2 F_{ij}^a \hat{T}_a.$$

It is possible to verify that

$$\begin{aligned} [\hat{L}_{ik}, x_j] &= i\delta_{ij} x_k - i\delta_{kj} x_i, \quad [\hat{L}_{ik}, \hat{\pi}_j] = i\delta_{ij} \hat{\pi}_k - i\delta_{kj} \hat{\pi}_i, \\ [\hat{L}_{ij}, \hat{L}_{mn}] &= i\delta_{im} \hat{L}_{jn} - i\delta_{jm} \hat{L}_{in} - i\delta_{in} \hat{L}_{jm} + i\delta_{jn} \hat{L}_{im}, \end{aligned}$$

i.e. \hat{L}_{ij} are indeed the generators of the group $SO(5)$ and $[\hat{H}, \hat{L}_{ij}] = 0$.

After some calculations, we have $[\hat{H}, \hat{M}_i] = 0$ which means that \hat{M}_i is the fact analog of the Runge-Lenz vector for Yang-Coulomb monopole. It can also be shown that

$$[\hat{L}_{ij}, \hat{M}_k] = i\delta_{ik} \hat{M}_j - i\delta_{jk} \hat{M}_i, \quad [\hat{M}_i, \hat{M}_k] = -2i\hat{H} \hat{L}_{ik}.$$

These commutation rules generalize relations known from the theory of the Coulomb problem [14].

Finally, let us introduce the 6×6 matrix

$$\hat{D} = \begin{pmatrix} \hat{L}_{ij} & -\hat{M}'_i \\ \hat{M}'_j & 0 \end{pmatrix},$$

where $\hat{M}'_i = (-2\hat{H})^{-1/2} \hat{M}_i$. The components $\hat{D}_{\mu\nu}$ (where $\mu, \nu = 0, 1, 2, 3, 4, 5$) satisfy the commutation relations

$$[\hat{D}_{\mu\nu}, \hat{D}_{\lambda\rho}] = i\delta_{\mu\lambda} \hat{D}_{\nu\rho} - i\delta_{\nu\lambda} \hat{D}_{\mu\rho} - i\delta_{\mu\rho} \hat{D}_{\nu\lambda} + i\delta_{\nu\rho} \hat{D}_{\mu\lambda},$$

i.e. $\hat{D}_{\mu\nu}$ are the generators of the group $SO(6)$. Since $[\hat{H}, \hat{D}_{\mu\nu}] = 0$, one concludes that Yang-Coulomb monopole is provided by the $SO(6)$ group of hidden symmetry.

For the continuous spectrum ($\epsilon > 0$) we have

$$[\hat{\tilde{M}}_i, \hat{\tilde{M}}_k] = -i\hat{L}_{ik},$$

where $\hat{\tilde{M}}_i = (2\hat{H})^{-1/2} \hat{M}_i$, and the group of the hidden symmetry is $SO(5, 1)$.

8 Energy Levels

The Casimir operators for $SO(6)$ are [15]

$$\begin{aligned}\hat{C}_2 &= \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\mu\nu}, \\ \hat{C}_3 &= \epsilon_{\mu\nu\rho\sigma\tau\lambda}\hat{D}_{\mu\nu}\hat{D}_{\rho\sigma}\hat{D}_{\tau\lambda}, \\ \hat{C}_4 &= \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\nu\rho}\hat{D}_{\rho\tau}\hat{D}_{\tau\mu}.\end{aligned}$$

According [16], the eigenvalues of these operators can be taken as

$$\begin{aligned}C_2 &= \mu_1(\mu_1 + 4) + \mu_2(\mu_2 + 2) + \mu_3^2, \\ C_3 &= 48(\mu_1 + 2)(\mu_2 + 1)\mu_3, \\ C_4 &= \mu_1^2(\mu_1 + 4)^2 + 6\mu_1(\mu_1 + 4) + \mu_2^2(\mu_2 + 2)^2 + \mu_3^4 - 2\mu_3^2,\end{aligned}$$

where μ_1 , μ_2 and μ_3 are positive integers or half-integers and $\mu_1 \geq \mu_2 \geq \mu_3$.

Direct calculation lead to the representation

$$\begin{aligned}\hat{C}_2 &= -\frac{e^4\mu_0}{2\hbar^2\hat{H}} + 2\hat{T}^2 - 4, \\ \hat{C}_3 &= 48\left(-\frac{\mu_0e^4}{2\hbar^2\hat{H}}\right)^{1/2}\hat{T}^2, \\ \hat{C}_4 &= \hat{C}_2^2 + 6\hat{C}_2 - 4\hat{C}_2\hat{T}^2 - 12\hat{T}^2 + 6\hat{T}^4.\end{aligned}\tag{13}$$

From the equation we can obtain another expression for the eigenvalue C_4

$$C_4 = [C_2 - 2T(T + 1)]^2 + 6[C_2 - 2T(T + 1)] + 2T^2(T + 1)^2$$

and calculate that

$$C_2 - 2T(T + 1) = \mu_1(\mu_1 + 4),\tag{14}$$

$$\mu_2^2(\mu_2 + 2)^2 + \mu_3^4 - 2\mu_3^2 = 2T^2(T + 1)^2.\tag{15}$$

The energy levels of the Yang-Coulomb monopole can be derived from (13) and (14)

$$\epsilon_N^T = -\frac{\mu_0e^4}{2\hbar^2(\frac{N}{2} + 2)^2},\tag{16}$$

where $\mu_1 = N/2$ and N -nonnegative integer number. The substitution of the eigenvalues of \hat{H} and \hat{T}^2 in the equation for \hat{C}_3 gives one more formula for C_3

$$C_3 = 48(\mu_1 + 2)T(T + 1).$$

Now, we have two expressions for C_3 and the comparison leads the relation

$$T(T+1) = (\mu_2 + 1)\mu_3. \quad (17)$$

Comparing (17) with (14) we will get the equation

$$(\mu_2^2 - \mu_3^2) [(\mu_2 + 2)^2 - \mu_3^2] = 0.$$

Since $\mu_3 \leq \mu_2$, one concludes that $\mu_3 = \mu_2$. Then, from (17) it follows that $\mu_2 = T$. Therefore, N in the formula (16) takes only values $\frac{N}{2} = T, T+1, T+2, \dots$, – the result known from our paper [17].

9 Conclusions

Formulae (5) and (6) together with the ansatz (3) form the duality transformation mapping of the eight-dimensional quantum oscillator into the charge-dyon system with the $SU(2)$ Yang monopole. This type of duality is valid not only for the 8D, 4D and 2D oscillators, but also for the oscillator-like systems with the potentials

$$V(u^2) = c_0 + c_1 u^2 + W(u^2),$$

where $W(u^2)$ has a polynomial form

$$W(u^2) = \sum_{n=2}^{\infty} c_n u^{2n}.$$

For such modified potentials, the ansatz (3) can be rewritten as

$$\epsilon = -\frac{c_1}{4}, \quad e^2 = \frac{E - c_0}{4}.$$

The hidden symmetry of the $SU(2)$ Yang-Coulomb monopole makes possible to solve the equation (7) in the five-dimensional hyperspherical, parabolic and elliptic coordinates by the separation of variables method [17, 18, 19].

Acknowledgments

I am very grateful to organizers who have invited me to this very interesting Workshop. This work was partially supported by the NATO Collaborative Linkage Grant No. 978431 and ANSEF Grant No. PS81.

10 Appendix

1. The Cartesian components of the gauge field tensor F_{ij}^a :

$$\begin{aligned} F_{01}^1 &= -\frac{x_4}{r^3}, & F_{02}^1 &= -\frac{x_3}{r^3}, & F_{03}^1 &= \frac{x_2}{r^3}, & F_{04}^1 &= \frac{x_1}{r^3}, \\ F_{12}^1 &= \frac{x_2 x_4 - x_1 x_3}{r^3(r+x_0)}, & F_{13}^1 &= \frac{x_1 x_2 + x_3 x_4}{r^3(r+x_0)}, & F_{14}^1 &= \frac{1}{r^2} \left[\frac{x_1^2 + x_4^2}{r(r+x_0)} - 1 \right], \\ F_{23}^1 &= \frac{1}{r^2} \left[\frac{x_2^2 + x_3^2}{r(r+x_0)} - 1 \right], & F_{24}^1 &= \frac{x_1 x_2 + x_3 x_4}{r^3(r+x_0)}, & F_{34}^1 &= -\frac{x_1 x_2 + x_3 x_4}{r^3(r+x_0)}. \end{aligned}$$

$$\begin{aligned}
F_{01}^2 &= -\frac{x_3}{r^3}, & F_{02}^2 &= \frac{x_4}{r^3}, & F_{03}^2 &= -\frac{x_1}{r^3}, & F_{04}^2 &= \frac{x_2}{r^3}, \\
F_{12}^2 &= -\frac{x_1x_4 + x_2x_3}{r^3(r+x_0)}, & F_{13}^2 &= -\frac{1}{r^2} \left[\frac{x_1^2 + x_3^2}{r(r+x_0)} - 1 \right], & F_{14}^2 &= \frac{x_1x_2 - x_3x_4}{r^3(r+x_0)}, \\
F_{23}^2 &= \frac{x_3x_4 - x_1x_2}{r^3(r+x_0)}, & F_{24}^2 &= \frac{1}{r^2} \left[\frac{x_2^2 + x_4^2}{r(r+x_0)} - 1 \right], & F_{34}^2 &= \frac{x_1x_4 + x_2x_3}{r^3(r+x_0)}.
\end{aligned}$$

$$\begin{aligned}
F_{01}^3 &= -\frac{x_2}{r^3}, & F_{02}^3 &= \frac{x_1}{r^3}, & F_{03}^3 &= -\frac{x_4}{r^3}, & F_{04}^3 &= \frac{x_3}{r^3}, \\
F_{12}^3 &= \frac{1}{r^2} \left[\frac{x_1^2 + x_2^2}{r(r+x_0)} - 1 \right], & F_{13}^3 &= \frac{x_2x_3 - x_1x_4}{r^3(r+x_0)}, & F_{14}^3 &= \frac{x_1x_3 + x_2x_4}{r^3(r+x_0)}, \\
F_{23}^3 &= -\frac{x_1x_3 + x_2x_4}{r^3(r+x_0)}, & F_{24}^3 &= \frac{x_2x_3 - x_1x_4}{r^3(r+x_0)}, & F_{34}^3 &= \frac{1}{r^2} \left[\frac{x_3^2 + x_4^2}{r(r+x_0)} - 1 \right].
\end{aligned}$$

2. The hyperspherical components of the gauge field tensor F_{ij}^a :

$$\begin{aligned}
F_{\theta\beta}^1 &= \frac{1}{2} \sin \theta \sin \alpha, & F_{\theta\alpha}^1 &= 0, & F_{\theta\gamma}^1 &= -\frac{1}{2} \sin \theta \sin \beta \cos \alpha, \\
F_{\beta\alpha}^1 &= -\frac{1}{4} \sin^2 \theta \cos \alpha, & F_{\beta\gamma}^1 &= -\frac{1}{4} \sin^2 \theta \cos \beta \cos \alpha, & F_{\alpha\gamma}^1 &= \frac{1}{4} \sin^2 \theta \sin \beta \sin \alpha.
\end{aligned}$$

$$\begin{aligned}
F_{\theta\beta}^2 &= \frac{1}{2} \sin \theta \cos \alpha, & F_{\theta\alpha}^2 &= 0, & F_{\theta\gamma}^2 &= \frac{1}{2} \sin \theta \sin \beta \sin \alpha, \\
F_{\beta\alpha}^2 &= \frac{1}{4} \sin^2 \theta \sin \alpha, & F_{\beta\gamma}^2 &= \frac{1}{4} \sin^2 \theta \cos \beta \sin \alpha, & F_{\alpha\gamma}^2 &= \frac{1}{4} \sin^2 \theta \sin \beta \cos \alpha.
\end{aligned}$$

$$\begin{aligned}
F_{\theta\beta}^3 &= 0, & F_{\theta\alpha}^3 &= \frac{1}{2} \sin \theta, & F_{\theta\gamma}^3 &= \frac{1}{2} \sin \theta \cos \beta, \\
F_{\beta\alpha}^3 &= 0, & F_{\beta\gamma}^3 &= -\frac{1}{4} \sin^2 \theta \sin \beta, & F_{\alpha\gamma}^3 &= 0.
\end{aligned}$$

References

- [1] J. Schwinger, Science, **165**, (1969), 757.
- [2] N.Seiberg, E.Witten. Nucl.Phys. **B431**, (1994), 484.
- [3] A. Hurwitz. Nurch. Ges. Wiss. Göttingen, 309, (1896).
- [4] T. Levi-Civita. Opera Mathematical **2**, (1956), 411.
- [5] P. Kustaanheimo, E. Stiefel. J. Reine Angew. Math. **218**, (1965), 204.

- [6] L.S.Davtyan, L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian, V.M.Ter-Antonyan. J. Phys. **A20**, (1987), 6121.
- [7] D. Lambert, M. Kibler. J. Phys. **A21**, (1988), 307.
- [8] M. Kibler, P. Winternitz. J. Phys. **A21**, (1988), 1787.
- [9] Le Van Hoang, Tony J.Viloria, Le anh Thu. J. Phys. **A24**, (1991), 3021.
- [10] L.G.Mardoyan, A.N.Sissakian, V.M.Ter-Antonyan. Oscillator as a Hidden Non-Abelian Monopole. Preprint JINR E2-96-24, Dubna, 1996. Hep-th/9601093.
- [11] C.N.Yang. J.Math.Phys., **19**, (1978), 320.
- [12] L.G.Mardoyan, A.N.Sissakian, V.M.Ter-Antonyan. Mod.Phys.Lett. **A14**, (1999), 1303.
- [13] G.Györgi, J.Revai. JETP, **48**, (1965), 1445.
- [14] L.D.Landau, E.M.Lifshitz. Quantum Mechanics (Pergamon Press, 1977).
- [15] A.O.Barut, R.Raczka. Theory of Group Representations and Applications (PWN-Polish Scientific Publishers, 1977).
- [16] A.M.Perelomov, V.S.Popov. JETP Lett., **2**, (1965), 34.
- [17] L.G.Mardoyan, A.N.Sissakian, V.M.Ter-Antonyan. Phys. Atom. Nucl., **61**, (1998), 1746.
- [18] L.G.Mardoyan, A.N.Sissakian, V.M.Ter-Antonyan. Theor. Math. Phys. **123**, (2000), 451.
- [19] L.G.Mardoyan. Phys. Atom. Nucl., **65**, (2002), 1063.